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Constants of Motion for Complex Hamiltonian with Sextic Potential in ECPS.

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ABSTRACT

We study the Hamiltonian with Sextic Potential and their generalizations incorporating additional rational functions. Objective of this article is to search for constants of motion for complex Hamiltonian with sextic potential system. In our discussion, we will make use of rationalization methods for complex dynamical systems on the extended complex phase plane (ECPS), for exactly solvable (ES) models. Invariants for such may be useful in the analysis of dynamical systems.

Key words: Sextic potential, Exact complex invariant, PT -symmetry

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INTRODUCTION

A complex (non-Hermitian) Hamiltonian H can provide real and bounded eigenvalues for certain domains of the underlying parameters if H is invariant under the simultaneous action of the space (P) and time (T) reversal. Complex Hamiltonian is no longer hermitian and ordinarily does not guarantee real eigenvalues, however, in its PT -symmetric form, the system is found to exhibit real eigenvalues [1, 2]. Now it is possible to study complex Hamiltonians (PT -symmetric) which were not considered earlier for not meeting the Hermiticity requirement Complex Hamiltonian have been discussed in many branches of physics, such as condensed matter physics, particle physics, plasma physics. For the complex crystal lattice whose potential is $V(x) = isinx$, while the Hamiltonian $H = p^2 + isinx$ is not hermitian, it is PT -symmetric [3] and all of its energy bands are real.

Then it is argued that the reality of the spectrum is a consequence of the combined action of parity and time reversal invariance of H. The parity operator P and the time reversal operator T are defined by their action on position and momentum operator (in quantum mechanics) as $P : x \to -x;\quad p \to -p;\quad T : x \to x;\quad p \to -p;\quad i \to -i.$ Complex version of x and p, written as $x = x_1 + ip_2;\ p = p_1 + ix_2,$ which have been used by Xavier and Aguiar [4] to develop an algorithm for the computation of the semi-classical coherent-state propagator. Kaushal [5] and his co-workers [6] has investigated the construction of complex invariants [7, 8] of 1D complex Hamiltonian systems on the extended complex phase plane (ECPS).

The organization of the paper is as follows: in section 2, the complexification of dynamical system and method of construction of integrals is described. In section 3, we apply the method to obtain a complex invariant of a dynamical system and finally concluding remarks are given in section 4.

THE RATIONALIZATION METHOD

Consider a one-dimensional real phase space (x, p), which may be transformed into a complex space (x_1, p_2, p_1, x_2), by defining position and momenta variables as

$$x = x_1 + ip_2;\quad p = p_1 + ix_2. \tag{1}$$

The presence of variables (p_2, x_2) in the above transformation (1), can be regarded as some sort of coordinate-momentum interaction of the dynamical system. From (1) one can easily obtain

$$\partial_x = \partial_{x_1} - i\partial_{p_2};\ \ \partial_p = \partial_{p_1} - i\partial_{x_2}, \tag{2}$$

For construction of exactly solvable (ES) models, consider a complex phase space function $I(x, p) = I_1 + iI_2$, as corresponding to $H(x, p) = H_1 + iH_2$ to be the TID (time-independent) dynamical invariant of the system in ECPS, then this must conform the following invariance condition.

$$d_t I = [I, H]_{PB} = 0, \tag{3}$$

where $[.,.]$ is the Poisson bracket (PB). Now using $I = I_1 + iI_2$, $H = H_1 + iH_2$ in (3), and after equating real and imaginary parts separately to zero, one obtains the following pair of equations: real part is

$$\begin{aligned}&(\partial_{x_1} I_1 + \partial_{p_2} I_2)(\partial_{p_1} H_1 + \partial_{x_2} H_2) - (\partial_{x_1} I_2 - \partial_{p_2} I_1)(\partial_{p_1} H_2 - \partial_{x_2} H_1)\\ &- (\partial_{p_1} I_1 + \partial_{x_2} I_2)(\partial_{x_1} H_1 + \partial_{p_2} H_2) + (\partial_{p_1} I_2 - \partial_{x_2} I_1)(\partial_{x_1} H_2 - \partial_{p_2} H_1) = 0.\end{aligned} \tag{4}$$

And imaginary part is

$$(\partial_{x_1} I_2 - \partial_{p_2} I_1)(\partial_{p_1} H_1 + \partial_{x_2} H_2) + (\partial_{x_1} I_1 + \partial_{p_2} I_2)(\partial_{p_1} H_2 - \partial_{x_2} H_1)$$
$$- (\partial_{p_1} I_2 - \partial_{x_2} I_1)(\partial_{x_1} H_1 + \partial_{p_2} H_2) - (\partial_{p_1} I_1 + \partial_{x_2} I_2)(\partial_{x_1} H_2 - \partial_{p_2} H_1) = 0. \quad (5)$$

For given $H(x, p, t)$ make an ansatz for I preferably in power of momentum p, using PT-symmetry both H and I reduce to the form $I = I_1 + iI_2$, $H = H_1 + iH_2$, and then substitute the resultant I_1, I_2, H_1 and H_2 in (4) and (5) and rationalize with respect to power of p_1 and x_2 and their combination will yield coupled partial differential equations for the arbitrary complex coefficient functions appearing in the ansatz for I. The substitution of solutions of these equations (if the solutions exits and are unique) in the ansatz for I then yield the final form of invariant. We make an ansatz for complex invariant I in the form

$$I = a_0(x) + a_2(x) p^2 \quad (6)$$

and write its complex version in the form $I = I_1 + iI_2,$ where

$$I_1 = a_{0r} + a_{2r}(p_1^2 - x_2^2) - 2a_{2i} p_1 x_2 \quad (7)$$

and

$$I_2 = a_{0i} + a_{0i}(p_1^2 - x_2^2) + 2a_{2r} p_1 x_2 \quad (8)$$

where the complex coefficient functions $a_k(x) = a_{kr}(x) + a_{ki}(x)$ are the real functions of their real arguments, that render I invariant are to be determined.

SEXTIC POTENTIAL

While the sextic potential has been studied thoroughly both from algebraic and analytic points of view, including PT-symmetry [9], no systematic study in an algebraic framework included for an invariant. Here an attempt is made to accomplish this. We discover that a rather general sextic potential with a barrier of the form $\frac{x^6}{6}$. Let us start considering a PT-symmetric potential:

$$H = \frac{p^2}{2} + \frac{\delta_1}{2} x^2 + \frac{\delta_2}{4} x^4 + \frac{\delta_3}{6} x^6 \quad (9)$$

Here we demonstrate that the complex version of (9), namely the PT-symmetric one obtained by using (1) in (9), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2) + \frac{\delta_1}{2}(x_1^2 - p_2^2) + \frac{\delta_2}{4}(x_1^4 + p_2^4 - 6x_1^2 p_2^2) + \frac{\delta_3}{6}(x_1^6 - 15x_1^4 + 15x_1^2 p_2^4 - p_2^6),$$
$$H_2 = p_1 x_2 + \delta_1 x_1 p_2 + \delta_2 (p_2 x_1^3 - x_1 p_2^3) + \delta_3 (x_1^5 p_2 + x_1 p_2^5 - \frac{10 x_1^3 p_2^3}{3}) \quad (10)$$

H_1 and H_2 are linearly independent with respect to the canonical pairs $(x_1, p_1), (x_2, p_2), (x_3, p_3)$ and (x_4, p_4). Substitution of (7), (8), (10) in (4) yields the expression

$$[\frac{\partial a_{0r}}{\partial x_1}+\frac{\partial a_{0i}}{\partial p_2}+(\frac{\partial a_{2r}}{\partial x_1}+\frac{\partial a_{2i}}{\partial p_2})(p_1^2-x_2^2)+2p_1x_2(\frac{\partial a_{2i}}{\partial x_1}-\frac{\partial a_{2r}}{\partial p_2})](2p_1)$$
$$-[\frac{\partial a_{0i}}{\partial x_1}-\frac{\partial a_{0r}}{\partial p_2}+(\frac{\partial a_{2i}}{\partial x_1}+\frac{\partial a_{2r}}{\partial p_2})(p_1^2-x_3^2)+2p_1x_3(\frac{\partial a_{2r}}{\partial x_1}+\frac{\partial a_{2i}}{\partial p_2})](2x_3)$$
$$-(p_1a_{2r}-x_2a_{2i})\{\delta_1x_1+\delta_2(x_1^3-3x_1p_2^2)+\delta_3(x_1^5-10x_1^3p_2^2+5x_1p_2^4)\}$$
$$-(p_1a_{2i}+x_2a_{2r})\{\delta_1p_2+\delta_2(-p_2^3+3x_1^2p_2)+\delta_3(5x_1^4p_2-10x_1^2p_2^3+p_2^5\}=0 \qquad (11)$$

which can be rationalized with respect to the power of p_1, x_2 and their combinations to give the following set of four coupled partial differential equations

$$\partial_{x_1}a_{0r}+\partial_{p_2}a_{0i}-4a_{2r}\{\delta_1x_1+\delta_2(x_1^3-3x_1p_2^2)+\delta_3(x_1^5-10x_1^3p_2^2+5x_1p_2^4)\}$$
$$+4a_{2i}\{\delta_1p_2+\delta_2(-p_2^3+3x_1^2p_2)+\delta_3(5x_1^4p_2-10x_1^2p_2^3+p_2^5\}=0, \qquad (12)$$

$$\partial_{p_2}a_{0r}-\partial_{x_1}a_{0i}+4a_{2i}\{\delta_1x_1+\delta_2(x_1^3-3x_1p_2^2)+\delta_3(x_1^5-10x_1^3p_2^2+5x_1p_2^4)\}$$
$$+4a_{2r}\{\delta_1p_2+\delta_2(-p_2^3+3x_1^2p_2)+\delta_3(5x_1^4p_2-10x_1^2p_2^3+p_2^5\}=0, \qquad (13)$$

$$-\partial_{x_1}a_{02xr}+\partial_{p_3}a_{02xi}=0, \qquad (14)$$

$$\partial_{p_3}a_{11xr}+\partial_{x_1}a_{11xi}=0. \qquad (15)$$

So for construction of complex invariants one has to find out solutions for following unknown parameters $a_{2i}(x), a_{2r}(x), a_{0i}(x), a_{0r}(x)$ which are functions of (x_1, p_2).

(A) Solutions for a_{2r}, a_{2i}: equations (14) and (15) can be reduced to similar second-order forms for the functions a_{2r}, a_{2i} respectively,as

$$\partial^2_{x_1^2}a_{2r}+\partial^2_{p_2^2}a_{2r}=0, \quad \partial^2_{x_1^2}a_{2i}+\partial^2_{p_2^2}a_{2i}=0 \qquad (16)$$

solution of (16) in the form

$$a_{2r}=\frac{\alpha}{2}(x_1^2-p_2^2)+\alpha_1x_1+\alpha_2p_3+\delta_1;\ a_{2i}=\frac{\beta}{2}(x_1^2-p_2^2)+\beta_1x_1+\beta_2p_3+\delta_2 \qquad (17)$$

where $\alpha,\beta,\alpha_1,\alpha_2,\beta_1,\beta_2,\delta_1,\delta_2$ are arbitrary constant of integration to be determined later.

(B) Similarly to solve a_{0r}, a_{0i}. on differentiating (12) with respect to x_1 and (13) with respect to p_2 and add the resultant expression

$$\partial^2_{x_1^2}a_{0r}+\partial^2_{p_2^2}a_{0r}=-8[\{\delta_1x_1+\delta_2(x_1^3-3x_1p_2^2)+\delta_3(x_1^5-10x_1^3p_2^2+5x_1p_2^4)\}\partial_{p_2}a_{2r}$$
$$+\{\delta_1p_2+\delta_2(-p_2^3+3x_1^2p_2)+\delta_3(5x_1^4p_2-10x_1^2p_2^3+p_2^5\}\partial_{x_1}a_{2i}]$$
$$=-8\{\beta_2\delta_1x_1+\beta_2\delta_2(x_1^3-3x_1p_2^2+\beta_2\delta_3(x_1^5-10x_1^3p_2^2+5x_1p_2^4)$$
$$+\beta_1\delta_1p_2+\beta_1\delta_2(3x_1^2p_2-p_2^3)+\beta_1\delta_3(5x_1^4p_2-10x_1^2p_2^3+p_2^5)\}$$
$$=-8\{\delta_1(\beta_2x_1+\beta_1p_2)+\delta_2(\beta_2x_1^3-\beta_1p_2^3-3\beta_2x_1p_2^2+3\beta_1x_1^2p_2)$$

$$+\delta_3(\beta_2 x_1^5+\beta_1 p_2^5-10\beta_2 x_1^3 p_2^2-10\beta_1 x_1^2 p_2^3+5\beta_2 x_1 p_2^4+5\beta_1 x_1^4 p_2)\} \qquad (18)$$

where we have used (14) and (15) and then expression (17) to simplify the right hand side. Solution immediately will yield

$$a_{0r}(x_1,p_2)=\frac{-4\delta_1}{3}(\beta_2 x_1^3+\beta_1 p_3^3)-\frac{2\delta_2}{5}(\beta_2 x_1^5-\beta_1 p_2^5)+\frac{\delta_2}{2}(\beta_2 x_1 p_2^4-\beta_1 x_1^4 p_2)$$
$$-\frac{4\delta_3}{21}(\beta_2 x_1^7+\beta_1 p_2^7)+4\delta_3(\beta_2 x_1^5 p_2^2+\beta_1 x_1^2 p_2^5)-\frac{4\delta_3}{3}(\beta_1 x_1^6 p_2+\beta_2 x_1 p_2^6) \qquad (19)$$

Another result for a_{01xr} can be obtained if one retains the term $(\frac{\partial a_{02xr}}{\partial x_1})$ and $(\frac{\partial a_{02xr}}{\partial p_3})$ in (18) instead of $(\frac{\partial a_{02xi}}{\partial p_3})$ and $(\frac{\partial a_{02xi}}{\partial x_1})$ and looks for the solution of the resultant PDE's again in the separable in the form x_1 and p_3. The corresponding result for a_{02xr} will yield some constraint relation, that would require that

$$\alpha=\beta=0,\ \alpha_2=\beta_1,\ \alpha_1=-\beta_2 \qquad (20)$$

With this choice now, arbitrary constants for the expressions a_{02xr} and a_{02xi} takes the form

$$a_{2r}=-\beta_2 x_1+\beta_1 p_2+\delta_1,$$
$$a_{2i}=+\beta_1 x_1+\beta_2 p_2+\delta_2 \qquad (21)$$

Same procedure as followed for a_{01xr} and solution in the form

$$a_{0i}(x_1,p_2)=\frac{4\delta_1}{3}(\beta_1 x_1^3-\beta_2 p_2^3)+\frac{2\delta_2}{5}(\beta_1 x_1^5+\beta_2 p_2^5)-\frac{\delta_2}{2}(\beta_1 x_1 p_2^4+\beta_2 x_1^4 p_2)$$
$$+\frac{4\delta_3}{21}(\beta_1 x_1^7-\beta_2 p_2^7)-4\delta_3(\beta_1 x_1^5 p_2^2-\beta_2 x_1^2 p_2^5)-\frac{4\delta_3}{3}(\beta_2 x_1^6 p_2-\beta_1 x_1 p_2^6) \qquad (22)$$

Note that the forms $a_{2i}(x), a_{2r}(x), a_{0i}(x), a_{0r}(x)$ from [(17)-(22)] are determined only from (4). With this expressions for the coefficient function when (5) is rationalized, one obtained several constrains so obtained are all $\delta's=0$.

Construction of Invariants. For the construction of complex invariants using the results [(17)-(22)] for $a_{2i}(x), a_{2r}(x), a_{0i}(x), a_{0r}(x)$ one can obtain the real and imaginary parts I_1 and I_2 from (7) and (8),

$$I_1=\frac{-4\delta_1}{3}(\beta_2 x_1^3+\beta_1 p_3^3)-\frac{2\delta_2}{5}(\beta_2 x_1^5-\beta_1 p_2^5)+\frac{\delta_2}{2}(\beta_2 x_1 p_2^4-\beta_1 x_1^4 p_2)$$
$$-\frac{4\delta_3}{21}(\beta_2 x_1^7+\beta_1 p_2^7)+4\delta_3(\beta_2 x_1^5 p_2^2+\beta_1 x_1^2 p_2^5)-\frac{4\delta_3}{3}(\beta_1 x_1^6 p_2+\beta_2 x_1 p_2^6)$$
$$+(-\beta_2 x_1+\beta_1 p_2)(p_1^2-x_2^2)-2p_1 x_2(\beta_1 x_1+\beta_2 p_2)$$

$$I_2=\frac{4\delta_1}{3}(\beta_1 x_1^3-\beta_2 p_2^3)+\frac{2\delta_2}{5}(\beta_1 x_1^5+\beta_2 p_2^5)-\frac{\delta_2}{2}(\beta_1 x_1 p_2^4+\beta_2 x_1^4 p_2)$$
$$+\frac{4\delta_3}{21}(\beta_1 x_1^7-\beta_2 p_2^7)-4\delta_3(\beta_1 x_1^5 p_2^2-\beta_2 x_1^2 p_2^5)-\frac{4\delta_3}{3}(\beta_2 x_1^6 p_2-\beta_1 x_1 p_2^6)$$

$$+(\beta_1 x_1+\beta_2 p_2)(p_1^2-x_2^2)+2p_1x_2(-\beta_2 x_1+\beta_1 p_2) \qquad (23)$$

and finally complex invariant I given by $I = I_1 + iI_2$ can be written as

$$I=\frac{4\delta_1 b}{3}(x_1^3+ip_2^3)+\frac{2\delta_2 b}{5}(x_1^5-ip_2^5)-\frac{\delta_2 b}{2}(x_1p_2^4-ix_1^4p_2)-\frac{4\delta_3 b}{21}(x_1^7+ip_2^7)$$
$$-4\delta_3 b(x_1^5p_2^2+ix_1^2p_2^5)+\frac{4\delta_3 b}{3}(ix_1^6p_2+x_1p_2^6)+(p_1^2-x_2^2+2p_1x_2)b(x_1-ip_2) \qquad (24)$$

where $b=-\beta_2+i\beta_1$ are arbitrary constant, which conforms the condition (4) in view of the PB.

CONCLUSION

As pointed out in section 1, construction of invariant for a dynamical system and its physical interpretation(s) for better theoretical understanding of a given phenomenon is active area of research . Only availability of a few or all [2, 5, 8, 9] invariants for a dynamical system definitely offers insight into the finer details as far as an understanding of the phenomenon is concerned. Finally, a few remarks about the applicability of the systems investigated in this work are in order. The role of a linear invariant designed, however, for a rotating TD harmonic oscillator in N -dimensions is investigated by [10] in the context of coherent states.

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