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About The Optimal Recovery Of The Derivatives Of Analytic Functions Defined In The Upper Half-Plane From Their Values At A Finite Number Of Points.

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ABSTRACT

In this article the author studies the problem of best approximation of a derivative at a point from a bounded analytic function defined in the upper half-plane. The problem is solved using information about the value of the function at the same point, as well as the values of the function in some finite set of different points. The article consists of an introduction and two sections. In the introduction, some concepts and results from K.Yu. Osipenko's article are given and the problem of optimal recovery of the derivative is formulated.In the first section he finds the error of the best approximation method, as well as the corresponding extremal function. In the second section the coefficients of the linear best approximation method are calculated.
Keywords: optimal recovery, error of the best method, extremal function, linear best method, coefficients of the linear best method.

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INTRODUCTION

Approximate calculations are often used to evaluate chemical and biological processes. Optimal function recovery is one of the methods of approximate calculations.Here I study the problem of optimal recovery of a derivative of an analytic function defined in the upper half-plane from values of the function in a finite number of points. First, let me recall some concepts and results from the work of K.Yu. Osipenko. Let X be a linear normed space, and W a unit sphere. Further, continuous linear functionals defined in X are denoted by $L, l_1, \ldots, l_n$. If $S(t_1, \ldots, t_n)$ is any complex function of n complex variables, then the error of approximation by the method S of the functional L from the values on the set W of functionals $l_1, \ldots, l_n$ is the following quantity:

$$r_n(S) = \sup_{x \in W} \left| L(x) - S\big(l_1(x), \ldots, l_n(x)\big) \right|$$

The method $S_0(t_1, \ldots, t_n)$ is called the best approximation (or recovery) method if the following equality holds:

$$r_n(S_0) = \inf_S r_n(S)$$

In [2] we proved that in this case there exists a linear best approximation method $S_0 = \sum_{k=1}^{n} c_k l_k(x)$ (here c_k are some complex numbers, the coefficients of the linear best method). In the above mentioned article Osipenko found, that the error of the best method is determined by the following formula:

$$r_n(S_0) = \sup_{\substack{x \in W \\ l_1(x) = \cdots = l_n(x) = 0}} |L(x)|. \qquad (1)$$

The upper half-plane is denoted by $D = \{z : Imz > 0\}$, and $B^1(D) = \{f(z) : |f(z)| \le 1\}$ is a family of analytic functions in D. Further, let $z_0, z_1, \ldots, z_n$ -be distinct points in D ($z_k = x_k + iy_k; k = 1, \ldots, n$).Consider the concrete functinality:

$$L(f) = f'(z_0), l_1(f) = f(z_1), \ldots, l_n(f) = f(z_n), l_{n+1}(f) = f(z_0);$$

here $f(z)$ is any bounded analytic function in D. According to the above mentioned facts, there is a linear best approximation method $c_0 f(z_0) + \sum_{k=1}^{n} c_k f(z_k)$ (c_k- coefficients of the linear best method; $k = 0, 1, \ldots, n$). The error of the best approximation method (denote it by $r_1(z_0, z_1, \ldots, z_n)$) is calculated by the following formula (see (1)):

$$r_1(z_0, z_1, \ldots, z_n) = \sup_{\substack{f(z) \in B^1(D) \\ f(z_1) = \ldots = f(z_n) = f(z_0) = 0}} |f'(z_0)| \qquad (2)$$

I would like to note that the problems of optimal recovery have been considered in many worksby Micchelli, Rivlin, Ovchintsev, Kusis (see [1], [2], [3]). And to find the coefficients of the linear best method (in the second section) I will need the following integral (see [4]):

$$\int_{-\infty}^{+\infty} \frac{dx}{|x - z_0|^2} = \frac{\pi}{y_0} \qquad (3)$$

where $z_0 = x_0 + iy_0 \ (y_0 > 0)$.

Finding the error of the best method

Denote by

$$B(z) = \prod_{k=1}^{n} \frac{z - z_k}{z - \overline{z_k}} \qquad (4)$$

the finite Blaschke product in the upper half-plane, and

$$A = \{f(z), f(z) \in B^1(D): f(z_1) = \cdots = f(z_n) = f(z_0) = 0\}$$

is a family of analytic functions. In order to find the error of the best method, we factorize the family of functions A. It will be more convenient for me to do this with the aid of a similar family of analytic functions in the unit circle. Denote by $K = \{\omega: |\omega| < 1\}, B^1(K) = \{g(\omega): |g(\omega)| \leq 1\}$.Let's pretend that $b \in D(b$-fixed;$Imb > 0)$.Let us study the mapping of the upper half-plane D onto the unit circle K:

$$\omega = F(z) = \frac{z-b}{z-\bar{b}}$$

I denote by $\omega_0 = F(z_0), \omega_1 = F(z_1), \ldots, \omega_n = F(z_n), A_1 = \{g(\omega): g(\omega) \in B^1(K), g(\omega_0) = g(\omega_1) = \cdots = g(\omega_n) = 0\}$, where

$$B_1(\omega) = \frac{\omega - \omega_0}{1 - \overline{\omega_0}\,\omega} \prod_{k=1}^{n} \frac{\omega - \omega_k}{1 - \overline{\omega_k}\,\omega}$$

is a finite Blaschke product in the circle K (we recall that if $|\omega| = 1$, then

$$|B_1(\omega)| = 1).$$

If $g(\omega) \in A_1$ then the function $\varphi(\omega)$ takes the following form:

$$\varphi(\omega) = \frac{g(\omega)}{B_1(\omega)} \in B^1(K)$$

Therefore $g(\omega) = B_1(\omega)\varphi(\omega)$, where $\varphi(\omega) \in B^1(K)$.

Now suppose the following: $f(z) \in A$.Consider the following function:

$$g(\omega) = f\left(\frac{b - \bar{b}\omega}{1 - \omega}\right)$$

It's obvious that $g(\omega_0) = g(\omega_1) = .. g(\omega_n) = 0$ and $g(\omega) \in B^1(K)$.This implies the following relation: $g(\omega) \in A_1$.Consequently:

$$g(\omega) = \frac{\omega - \omega_0}{1 - \overline{\omega_0}\,\omega} \prod_{k=1}^{n} \frac{\omega - \omega_k}{1 - \overline{\omega_k}\,\omega} \varphi(\omega) \qquad (5)$$

Since

$$\frac{\omega - \omega_k}{1 - \overline{\omega_k}\omega} = \frac{\left(\frac{z-b}{z-\bar{b}} - \frac{z_k - b}{z_k - \bar{b}}\right)}{\left(1 - \frac{\overline{z_k} - \bar{b}}{\overline{z_k} - b}\frac{z-b}{z-\bar{b}}\right)} = -\frac{\overline{z_k} - b}{z_k - \bar{b}} \frac{z - z_k}{z - \overline{z_k}},$$

then (see (5))

$$f(z) = \frac{z - z_0}{z - \overline{z_0}} \prod_{k=1}^{n} \frac{z - z_k}{z - \overline{z_k}} h(z),$$

Where

$$h(z) \in B^1(D).$$

Then

$$f'(z_0) = h(z_0)B(z_0)\frac{1}{2iy_0}.$$

This implies the equality:

$$r_1(z_0, z_1, \dots, z_n) = \frac{|B(z_0)|}{2y_0} \sup_{h(z)\in B^1(D)} |h(z_0)| = \frac{|B(z_0)|}{2y_0} \qquad (6)$$

Note. The extremal function of problem (2) is unique up to a factor $e^{i\delta}$, in which δ —is a real constant, which has the next form:

$$f^*(z) = e^{i\delta}\frac{z - z_0}{z - \bar{z}_0}B(z)$$

Calculating the coefficients of the linear best recovery method

It follows from [4] that the linear best approximation method is unique. In order to find its coefficients, we apply the following integral:

$$J = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{B(z_0)}{B(x)}\frac{f(x)}{(x - z_0)^2}dx \qquad (7)$$

Here B(z)is determined by the formula (4), and $f(z) \in B^1(D)$.I calculate the integral (7) with the help of the residue theorem. For this, it is necessary to verify the fulfillment of the corresponding condition. Let introduce the following notation: $\gamma(R) = \{z: |z| = R, Im\, z \geq 0\}$ ($R > 0$; semicircle $\gamma(R)$ does not pass through points $z_0, z_1, \dots, z_n$),

$$\Phi(z) = \frac{B(z_0)}{B(z)}\frac{f(z)}{(z - z_0)^2} \quad (8)$$

$$Q(R) = \int_{\gamma(R)}\Phi(z)\,dz \qquad (9)$$

Let estimate the value Q(R) modulo:

$$|Q(R)| \leq |B(z_0)| \int_{\gamma(R)}\left|\prod_{k=1}^{n}\frac{(z - \bar{z}_k)}{(z - z_k)}\right| |f(z)| \frac{1}{|z - z_0|^2}|dz| \leq$$

$$\leq |B(z_0)| \int_{\gamma(R)}\left|\prod_{k=1}^{n}\frac{\left(1 - \frac{\bar{z}_k}{z}\right)}{\left(1 - \frac{z_k}{z}\right)}\right| \frac{1}{|z|^2}\frac{1}{\left|1 - \frac{z_0}{z}\right|^2}|dz|$$

Let introduce an additional notation:

$$q(z) = \prod_{k=1}^{n}\frac{\left(1 - \frac{\bar{z}_k}{z}\right)}{\left(1 - \frac{z_k}{z}\right)}\frac{1}{\left(1 - \frac{z_0}{z}\right)^2}$$

Since

$$\lim_{z\to\infty} q(z) = 1$$

then there exists a number $R_0 (R_0 > 0)$ such that

$|q(z)| \leq M$ for all z for which $|z| \geq R_0$ (M is a constant). This implies the inequality:

$$|Q(R)| \leq |B(z_0)|\, M \,\frac{1}{R^2} \int_{\gamma(R)} |dz| = \pi\, |B(z_0)|\, M \,\frac{1}{R}$$

Since

$$\lim_{R\to+\infty} \pi |B(z_0)|\, M\, \frac{1}{R} = 0$$

Then

$$\lim_{R\to+\infty} Q(R) = 0$$

and, therefore, it is possible to apply the residue theorem to the integral (7) (see (8) and (9)).

Note. The integral (7) converges in the usual sense. In fact, this follows from the inequality (see (7), (8)):

$$|\Phi(x)| \leq \frac{C}{x^2}$$

where C is a constant.

Singular points $z_0, z_1, \dots, z_n$ of the functions $\Phi(z)$ are poles. Let me find the residues at these points:

$$\operatorname*{res}_{z=z_0} \Phi(z) = B(z_0) \lim_{z\to z_0} \left(\frac{f(z)}{B(z)}\right)' = f'(z_0) - \frac{B'(z_0)}{B(z_0)} f(z_0)$$

$$\operatorname*{res}_{z=z_k} \Phi(z) = \lim_{z\to z_k} (z - z_k) \frac{B(z_0)}{B(z)} \frac{f(z)}{(z - z_0)^2} =$$

$$= B(z_0) \lim_{z\to z_k} \frac{z - z_k}{\left(\frac{z-z_k}{z-\bar{z}_k}\right) \prod_{\substack{l=1\\ l\neq k}}^{n} \frac{z-z_l}{z-\bar{z}_l} (z - z_0)^2} f(z) =$$

$$= \frac{B(z_0)(z_k - \bar{z}_k)\, f(z_k)}{\prod_{\substack{l=1\\ l\neq k}}^{n} \frac{z_k-z_l}{z_k-\bar{z}_l} (z_k - z_0)^2} = -\frac{2B(z_0) y_k\, f(z_k)}{i \prod_{\substack{l=1\\ l\neq k}}^{n} \frac{z_k-z_l}{z_k-\bar{z}_l} (z_k - z_0)^2}$$

Denote

$$c_0 = \frac{B'(z_0)}{B(z_0)} \qquad (10)$$

$$c_k = \frac{2B(z_0) y_k}{i \prod_{\substack{l=1\\ l\neq k}}^{n} \frac{z_k-z_l}{z_k-\bar{z}_l} (z_k - z_0)^2}, \qquad (k = 1, \dots, n) \qquad (11)$$

Then (see (7))

$$J = f'(z_0) - c_0 f(z_0) - \sum_{k=1}^{n} c_k f(z_k) \qquad (12)$$

Let estimate the integral J modulo from above.There is the following estimate (see(7) and (3)):

$$|J| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|B(z_0)|}{|B(x)|}\frac{|f(x)|}{|x-z_0|^2}\,dx \le \frac{|B(z_0)|}{2\pi}\int_{-\infty}^{+\infty}\frac{dx}{|x-z_0|^2} = \\ = \frac{|B(z_0)|}{2\pi}\frac{\pi}{y_0} = \frac{|B(z_0)|}{2y_0}$$

Thus, for all functions $f(z) \in B^1(D)$the following estimate holds (see (12) and (6)):

$$\left|f'(z_0) - c_0 f(z_0) - \sum_{k=1}^{n} c_k f(z_k)\right| \le r_1(z_0, z_1, \dots, z_n)$$

Hence it follows that the method

$$c_0 f(z_0) + \sum_{k=1}^{n} c_k\, f(z_k)$$

is the linear best approximation method. In this method, the coefficientsc_k $(k = 1, \dots, n)$ are calculated by the formulas (11), and the coefficient c_0 by formula (10).

CONCLUSION

Thus, the coefficients of the linear best approximation method are found and its error is calculated, i.e. the problem is solved/

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